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SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS USING ADOMIAN DECOMPOSITIOIN METHOD

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Abstract

Partial differential equations occur frequently in Science and Engineering. Analytical solutions are generally hard to obtain. In this paper, Adomian Decomposition Method [1] is applied to solve one and two dimensional heat [2] equations. The method decomposes the solution into convergent series. The solution if available in closed form, the series will converge to it. The solutions so obtained are represented graphically.

1. Introduction

Adomian Decomposition Method (ADM) was first introduced by George Adomian (1922-1996) in the 1980's for solving non- linear differential equations. Since then, this method has been known as the ADM. The technique is based on decomposition of a solution of a non-linear operator equation in a series of the unknown function. Each term of the series is obtained from a polynomial generated from an expansion of an analytic function into a power series. The technique is clear in the formulation but the difficulty arises in computing the Adomian polynomials for nonlinear terms and in proving the convergence of the series of solutions. The main advantage of the method

Key Words and Phrases : *Adomian Decomposition Method, Convergent series.*

is that it can be applied directly to all types of differential and integral equations, linear or non-linear, homogenous or inhomogeneous with constant coefficient or with variable coefficients. Another important advantage is that the method is capable of greatly reducing the size of computational work while still maintaining high accuracy of the numerical solution. This research paper gives a clear and simple numerical experiments to test its implementation to heat equations.

2. Methodology of Adomian Decomposition Method

Adomian Decmposition Method decomposes a solution of a differential equation into infinite series. The equation

$$F\left[x, y(x)\right] = 0 \tag{1}$$

can be separated into the components

$$L\left[y(x)\right] + N\left[y(x)\right] = 0 \tag{2}$$

where L and N are linear and nonlinear components of F respectively. Solving for $L[y(x)]$ gives

$$L\left[y(x)\right] = -N\left[y(x)\right] \tag{3}$$

By applying the inverse operator L^{-1} on (3) yields

$$y(x) = -L^{-1}\left[N\left[y(x)\right]\right] + \phi\left(x\right) \tag{4}$$

where $\phi\left(x\right)$ is a costant of integration satisfying $L\phi\left(x\right) = 0$. Assume the solution $y(x)$ can be expressed as an infinite series given by

$$y\left(x\right) = \sum_{n=0}^{\infty} y_n\left(x\right) \tag{5}$$

Suppose also that nonlinear term $N\left[y\left(x\right)\right]$ can be expressed in terms of Adomian Polynomials A_n of the form

$$N\left[y\left(x\right)\right] = \sum_{n=0}^{\infty} A_n \tag{6}$$

where A_n are evaluated using the formula due to [1].

$$A_n = \frac{1}{n!}\frac{d^n}{d\mu^n}\, N\left(\sum_{n=0}^{\infty}\left(\mu^n A_n\right)\right)\Bigg|_{\mu=0} \quad ; n = 0, 1, 2, ...$$

Substituting (5) and (6) into (4) gives

$$\sum_{n=0}^{\infty} y_n(x) = \phi(x) - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right) \tag{7}$$

The recurrence relation from (7) is obtained as;

$$y_0 = \mu(x)\, y_{n+1} - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right); n \geq 0. \tag{8}$$

3. Numerical Experiments

Experiment 1

Use Adomian Decomposition Method to solve the initial-boundary value problem.

$$\begin{aligned} &u_t = u_{xx}, 0 < x < \pi, t > 0 \\ &u(0,t) = 0, u(\pi,t) = 0, t \geq 0 \\ &u(x,0) = \sin x \end{aligned} \tag{9}$$

Solution :

In an operator form, Equation (9) can be written as

$$L_t u(x,t) = L_x u(x,t) \tag{10}$$

Applying $L_t^{-1} = \int\limits_0^t (.)dt$ to both sides of (10) and using the initial condition we find

$$L_t^{-1}\left(L_t(u(x,t))\right) = L_t^{-1}(L_x(u(x,t))) \tag{11}$$

$$u(u,t) = \sin x + L_t^{-1}\left(L_x\left(u\left(x,t\right)\right)\right) \tag{12}$$

We next define the unknown function by a sum of components defined by the series and substitute in (12)

$$\sum_{n=0}^{\infty} u_n(x,t) = \sin x + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n(x,t\right)\right) \tag{13}$$

or equivalenty

$$u_0 + u_1 + u_2 + \cdots = \sin x + L_t^{-1}\left(L_x\left(u_0 + u_1 + u_2 + \cdots\right)\right). \tag{14}$$

Identifying the zeroth component $u_0(x,t)$ as assumed before and following the recursive algorithm (13) we obtain

$$\begin{aligned} &u_0(x,t) = \sin x \\ &u_1(x,t) = L_t^{-1}(L_x(u_0)) = -t\, \sin x, \\ &u_2(x,t) = L_t^{-1}(L_x(u_1)) = \tfrac{1}{2!}t^2\, \sin x, \\ &\vdots \end{aligned} \tag{15}$$

Consequently ,the solution $u(x,t)$ in a series form is given by

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \\ &= \sin x\left(1 - t + \tfrac{1}{2!}t^2 - \cdots\right), \end{aligned} \tag{16}$$

and in a closed form by

$$u(x,t) = e^{-t}\sin x \tag{17}$$

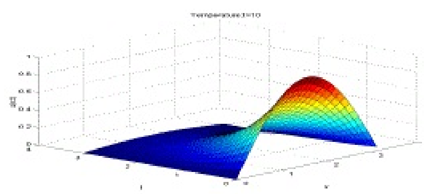

Figure 1: Temperature

Numerical Experiment 2

Use the Adomian decomposition method to solve the initial boundary value problem

$$\begin{aligned} &u_t = u_{xx} + u_{yy}, 0 < x, y < \pi, t > 0 \\ &u(0,y,t) = u(\pi,y,t) = 0 \\ &u(x,0,t) = u(x,\pi,t) = 0 \\ &u(x,y,0) = \sin x \sin y \end{aligned} \tag{18}$$

Solution:

We first write equation (18) in an operator form as

$$L_t u = L_x u + L_y u. \tag{19}$$

Applying the inverse operator L_t^{-1} to equation (19) and using the initial condition we obtain

$$u(x,y,t) = \sin x \sin y + L_t^{-1}(L_x u + L_y u). \tag{20}$$

where the components $u_n(x,y,t), n \geq 0$ are to be determined by using a recursive algorithm. Using ADM on equation (20) yields

$$\sum_{n=0}^{\infty} u_n = \sin x \sin y + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right)\right). \tag{21}$$

The zeroth component $u_0(x,y,t)$ is identified by all terms that are not included under. The components $u_n(x,y,t), n \geq 0$ can be completely determined by following the recursive algorithm

$$\begin{aligned} &u_0(x,y,t) = \sin x \sin y, \\ &u_{k+1}(x,y,t) = L_t^{-1}\left(L_x u_k + L_y u_k\right), \quad k \geq 0 \end{aligned} \tag{22}$$

With u_0 defined as shown above, the first few terms of the decomposition of equation (22) are given by

$$\begin{aligned} &u_0(x,y,t) = \sin x \sin y, \\ &u_1(x,y,t) = L_t^{-1}\left(L_x u_0 + L_y u_0\right) = -2t \sin x \sin y, \\ &u_2(x,y,t) = L_t^{-1}\left(L_x u_1 + L_y u_1\right) = \tfrac{(2t)^2}{2!} \sin x \sin y, \\ &u_3(x,y,t) = L_t^{-1}\left(L_x u_2 + L_y u_2\right) = -\tfrac{(2t)^3}{3!} \sin x \sin y, \end{aligned} \tag{23}$$

and so on. Combining the terms of (23), the solution in a series form is given by

$$u(x,y,t) = e^{-2t} \sin x \sin y. \tag{24}$$

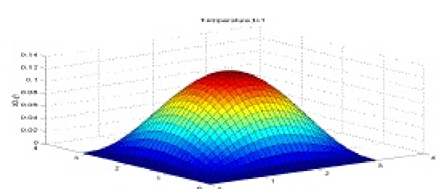

Figure 2: Temperature at t=10

4. Conclusion

The graphs are in excellent agreement with those obtained using numerical methods. ADM has the advantage in that the series converges very fast to a closed form solution by using fewer terms. No discretization is also required.

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